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Approximating Potential Integrals by Cardinal Basis Interpolants on Multivariate Scattered Data

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Abstract—A multivariate interpolation operator on scattered data, expressed as a convex combination of cardinal basis functions depending on the inverse $(s-2)$-power of the Euclidean distance in $\mathbb{R}^s$ $(s \geq 3)$ is proposed to give numerical approximations of the integral representing the potential function of the Newtonian field generated by a continuous mass distribution. The operator can be used to interpolate the mass density or directly the potential function, as well as to remap them on a regular grid or a convenient point set. Considerations on the Newtonian potential energy of a system of mass points permit us to introduce quite naturally the operator and to prove some remarkable properties; then the application to the continuous case is considered. Computational performances and possible applications of the operator are outlined. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords—Potential integrals, Scattered data, Multivariate interpolation, Cardinal basis, Parallel multistage and recursive procedures.

1. INTRODUCTION

The interpolation operator, which concerns us, falls within the following definition.

DEFINITION 1. *Let $S_n = \{u_j,\ j = 1,\ldots,n\}$ be a set of distinct points, in general irregularly distributed in a domain $D \subset \mathbb{R}^s$, for $s = 3,4,\ldots$, with associated real values $\{f_j,\ j = 1,\ldots,n\}$, and $Y(D)$ a linear space, spanned by continuous basis functions $g_j : D \to \mathbb{R}$, such that*

$$g_j(x) = \frac{\prod_{k=1,\,k\neq j}^{n} d^{s-2}(x,u_k)}{\sum_{i=1}^{n} \prod_{h=1,\,h\neq i}^{n} d^{s-2}(x,u_h)}, \qquad j = 1,2,\ldots,n,$$

where $d(x,u_k)$ is the Euclidean distance between any point $x \in \mathbb{R}^s$ and $u_k \in S_n$. Then, we define in $Y(D)$ the operator

$$\Phi(x;f,S_n) = \sum_{j=1}^{n} f(u_j) g_j(x) = \sum_{j=1}^{n} f(u_j) \frac{\prod_{k=1,\,k\neq j}^{n} d^{s-2}(x,u_k)}{\sum_{i=1}^{n} \prod_{h=1,\,h\neq i}^{n} d^{s-2}(x,u_h)},$$

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or equivalently,

$$\Phi(x; f, S_n) = \sum_{j=1}^{n} f(u_j) \frac{1/d^{s-2}(x, u_j)}{\sum_{i=1}^{n} (1/d^{s-2}(x, u_i))},$$
$$\Phi(u_j; f, S_n) \equiv \lim_{x \to u_j} \Phi(x; f, S_n) = f(u_j), \qquad j = 1, 2, \ldots, n. \tag{1}$$

The points u_j are the *nodes* and f_j are the *function values*. It is convenient to think of f_j as being generated by a function f, i.e., $f_j \equiv f(u_j)$, $j = 1, \ldots, n$.

We remark on some noteworthy properties enjoyed by $\Phi(x; f, S_n)$.

PROPERTY 2. *The basis functions satisfy the relations*

$$g_k(x) \geq 0, \qquad \sum_{k=1}^{n} g_k(x) = 1, \qquad g_k(u_i) = \delta_{ki},$$

where δ_{ki} is the Kronecker delta.

Hence, $\{g_k(x),\ k = 1, 2, \ldots, n\}$ is a set of cardinal functions in $Y(D)$. As a consequence, we have the following property.

PROPERTY 3. *The operator $\Phi(x; f, S_n)$ satisfies the interpolation conditions.*

PROPERTY 4. *The operator $\Phi(x; f, S_n)$ satisfies the characteristic properties of a weighted arithmetic mean*

(a) $\min_j f_j \leq \Phi(x; f, S_n) \leq \max_j f_j$;

(b) *if $f_j = c$, $j = 1, 2, \ldots, n$, where c is a constant, then $\Phi(x; c, S_n) \equiv c$.*

Formula (1) is a weighted average of the real function values $f(u_j)$, where the weighting is the inverse $(s-2)$-power of the Euclidean distance between x and u_j, and shows a deep similarity to a well-known operator for scattered multivariate interpolation attributed to Shepard [1], but probably used previously by others (see [2]). From the very beginning, Shepard's formula has been repeatedly connected with physical concepts, more or less explicitly. For example, Gordon and Wixom [3] wrote that Shepard's operator is essentially analogous to a gravitational law. Seeking clear connections between formula (1) and the physical world is important, in order to get more information on the properties and applicability of the formula itself (see [2,4] for some preliminary results).

In this paper, we propose a new application of the operator $\Phi(x; f, S_n)$, showing that it can be used to give numerical approximations of the multiple integral dependent on a parameter

$$U(x) = \frac{1}{s-2} \int_V \frac{\mu(u)}{d^{s-2}(x, u)}\, du, \qquad s > 2, \tag{2}$$

which represents the potential function of the Newtonian field generated by a continuous mass distribution in the domain V with mass density $\mu(u)$ (see, e.g., [5]). Since for $s = 3$ the field occurs according to Newton's law in $\mathbb{R}^3$, this explains the name of Newtonian potential given to the function $U(x)$. It is well known that, in general, the calculation of integral (2) is difficult, since the integrand function is often very complicated (for example, when the mass density is not uniform), or its values are known only at some points (as happens taking measurements).

Formula (1) can be applied to the evaluation of the potential function (2) at least in three ways, which are not quite equivalent and work in different situations.

(i) *Interpolating Density.* First, we affirm that the potential function (2) can be approximated as follows

$$U(x) \approx \sum_{j=1}^{n} \mu(u_j) \frac{1/d^{s-2}(x, u_j)}{\sum_{k=1}^{n} (1/d^{s-2}(x, u_k))} \frac{1}{s-2} \int_V \frac{1}{d^{s-2}(x, u)}\, du$$
$$= \Phi(x; \mu, S_n) \frac{1}{s-2} \int_V \frac{1}{d^{s-2}(x, u)}\, du, \tag{3}$$

provided n is sufficiently large and the distribution of the points u_j satisfies a soft condition (see Section 3). The last integral in (3) does not depend on the function μ and, in general, is easier to calculate than integral (2), because it corresponds to the most frequently considered case, namely that of a potential function with a constant mass density. However, only when the considered region is very simple, can the solution be found explicitly. These explicit solutions are usually given as series and, if one wishes to know the solution at a specific point, the series may require truncation and computer approximation.

(ii) *Interpolating Potential.* Another feature, which is no less important, arises using operator (1) to interpolate directly the potential function. Namely, if we suppose to know the potential values at a finite number of scattered points, then it is possible to build up the interpolant by means of these data and use it to get other potential values, as follows

$$U(x) \approx \Phi(x; U, S_n) = \sum_{j=1}^{n} U(u_j) \frac{1/d^{s-2}(x, u_j)}{\sum_{k=1}^{n} (1/d^{s-2}(x, u_k))}. \tag{4}$$

A relevant advantage of (4) is that it is numerically efficient. In particular, it works quite well even if the evaluation point x coincides with any node u_j. Moreover, choosing the evaluation point near to a node does not cause distortions of the potential. Among other things, the formula can be considered for approximation of gravity anomalies, on the analogy of the application of multiquadrics suggested by Hardy [6].

(iii) *Remapping Density or Potential.* Formula $\Phi(x; \mu, S_n)$ can also be used to calculate those values of $\mu(x)$, which are needed to use numerical integration formulas for the evaluation of the potential integral (2). In fact, these rules work usually on suitably collocated nodes or rectilinear grids. Similarly, in the representation of equipotential surfaces (see Section 4), $\Phi(x; U, S_n)$ can be employed to obtain values of $U(x)$ at the nodes of a rectilinear grid, since this is required by many standard plotting routines.

REMARK. It is well known that a point mass model of the form

$$U(x) \approx \sum_{j=1}^{n} \frac{\beta_j}{d^{s-2}(x, u_j)} \tag{5}$$

may be viewed as a numerical approximation of the potential integral (2) (see, e.g., [6,7]). Now, both approximations (3) and (4) can be formally considered from this point of view. In fact, they can be rewritten, respectively,

$$U(x) \approx \sum_{j=1}^{n} \left(\mu(u_j) \frac{1}{s-2} \frac{\int_V (1/d^{s-2}(x, u))\, du}{\sum_{k=1}^{n} (1/d^{s-2}(x, u_k))} \right) \frac{1}{d^{s-2}(x, u_j)},$$

and, supposing $U(x)$ continuous,

$$U(x) \approx \sum_{j=1}^{n} \left(\frac{1}{s-2} \frac{\int_V (\mu(u)/d^{s-2}(u_j, u))\, du}{\sum_{k=1}^{n} (1/d^{s-2}(x, u_k))} \right) \frac{1}{d^{s-2}(x, u_j)}.$$

These expressions give unusual meanings to the point masses β_j in (5) and show explicitly the dependence of β_j from the different quantities. A disadvantage of the point mass model (5) is that it fails completely if the evaluation point x coincides with any node u_j, making $d^{s-2}(x, u_j) = 0$. To avoid this drawback, Hardy [6] suggested another mass model, which works also if x coincides

with u_j and can be well interpolated by the multiquadric biharmonic method. Now, both formulas (3) and (4) work also for $x = u_j$ and give a performance comparable to that of Hardy's interpolant.

In Section 2, operator (1) is obtained by means of considerations on Newtonian potential energy of a system of material points. Further straightforward developments permit us to prove some remarkable properties of the operator. In Section 3, the case of the continuous mass distribution is considered as a natural extension of the discrete distribution case. Many of the considerations made for the latter are extended to the former; in particular, a proof is given for approximation (3). Section 4 deals with the approximation of equipotential surfaces and their graphical representation. To obtain smooth surfaces, it is necessary to modify $\Phi(x; f, S_n)$ in (1) considering, instead of the function values f_j, convenient local approximations to $f(x)$ at the nodes u_j. In Section 5, the computational performance of (3) and (4) is outlined. Actually, the numerical behavior of $\Phi(x; f, S_n)$ is well known from many applications (see, for instance, the references quoted in the section), while a full verification of an advantageous use of (3) and (4) in the evaluation of Newtonian potential still requires a lot of numerical tests in an interdisciplinary context. As a matter of fact, because of interdisciplinary aspects of potential theory and its applications (see, e.g., [8]), the considerations developed below on the Newtonian potential could be used as a research tool in specialized areas, such as solid mechanics of elasticity, fluid dynamics, geomagnetic fields, geothermal fields, diffusion processes, geodesy, velocity potential in atmospheric science, etc.

This paper is mainly a derivation of numerical methods, and a future detailed paper showing numerical results is in the process of being written.

2. THE INTERPOLATION OPERATOR FOR DISCRETE DISTRIBUTIONS

Operator (1) can be introduced by considerations on Newtonian potential energy relating to a system of mass points.

PROPERTY 5. *Let us consider in the Euclidean space $\mathbb{R}^s$ a system of mass points $f_j \equiv f(u_j)$, $j = 1, 2, \ldots, n$, placed at the points u_j, and a mass q at a point x, with $x \neq u_j$. The potential energy of the mass point q in the field generated by the discrete mass distribution*

$$W_d(x) = q \frac{1}{s-2} \sum_{j=1}^{n} f_j \frac{1}{d^{s-2}(x, u_j)}$$

does not change if each mass point f_j is replaced by a suitable mass $\Phi(x; f, S_n)$ given by the former relation in (1).

PROOF. In the quantity $W_d(x)$, each f_j is replaced by a suitable mass point c such that

$$q \frac{1}{s-2} \sum_{j=1}^{n} f_j \frac{1}{d^{s-2}(x, u_j)} = q \frac{1}{s-2} \sum_{j=1}^{n} c \frac{1}{d^{s-2}(x, u_j)}.$$

The equation means an invariance property, and from this, we obtain $c \equiv \Phi(x; f, S_n)$, given by the former relation in (1). The evaluation of $\Phi(x; f, S_n)$ at a point u_j leads to a division by zero; however, the definition of $\Phi(x; f, S_n)$ can be extended continuously as in the latter relation in (1). ∎

The weighted mean $\Phi(x; f, S_n)$ is a "summary" of the distribution $\{u_j, f_j\}$ and enjoys the characteristic property that it can substitute $\{u_j, f_j\}$ with another distribution $\{u_j, \Phi(x; f, S_n)\}$, yet discrete but uniform, without varying the value of the potential energy $W_d(x)$. On the other hand, $\Phi(x; f, S_n)$ is a real function of $x \in \mathbb{R}^s$, whose graph is a surface in $\mathbb{R}^s \times \mathbb{R}$, containing

the points (u_j, f_j), $j = 1, 2, \ldots, n$. This does not coincide, in general, with the surface associated to $f(x)$, but the former interpolates the latter at the nodes. Either leaving the physical model out of consideration or interpreting the values f_j as a sort of "masses", we can think of using (1) as an interpolation formula, which works well with a certain generality.

As a weighted mean $\Phi(x; f, S_n)$ fulfills a noteworthy minimum property, that is, the functional

$$\sum_{j=1}^{n} (f_j - y)^2 \frac{1}{d^{s-2}(x, u_j)}, \tag{6}$$

where y is a real number, is minimized by assuming $y = \Phi(x; f, S_n)$ (see, e.g., [2]). From a different point of view, minimizing the functional (6) can be interpreted as an approximation problem by weighted least squares method of the set of points $\{u_j, f_j\}$ by means of a hyperplane orthogonal to the f-axis [9].

The function $\Phi(x; f, S_n)$ is connected with the potential $U_d(x)$ of the discrete mass distribution by the relation

$$U_d(x) = \Phi(x; f, S_n) \frac{1}{s-2} \sum_{j=1}^{n} \frac{1}{d^{s-2}(x, u_j)}.$$

Here $\Phi(x; f, S_n)$ has the physical dimension of a Newtonian mass, while, in general, $\Phi(x; f, S_n)$ has the same physical dimension as the quantity f which is interpolated; for example, $\Phi(x; f, S_n)$ in (4) is a potential.

REMARK. We observe that quite similar results can be obtained considering a system of electrostatic charge points. Moreover, the connection between formula (1) and the electrostatic model (extended to $\mathbb{R}^s$) seems more natural than the connection with the Newtonian model, since the values f_j in (1) can now be either positive, null, or negative. Nevertheless, it is possible to get only positive function values, as occurs with gravitational masses, considering new values f_j^* defined as

$$f_j^* = f_j + b, \qquad \text{with } b > \max_j |f_j|, \quad j = 1, 2, \ldots, n.$$

Since the operator $\Phi(x; f, S_n)$ in (1) is linear and approximates constant functions exactly, we have

$$\Phi(x; f^*, S_n) = \Phi(x; f, S_n) + b,$$

so that we simply obtain a new surface, equal to the previous one but translated over the length b in the positive sense of the f-axis.

In the considered physical models, potential energies can be algebraically added, as a consequence of the force superposition principle. This property makes it possible to express operator (1) in the new form (7) below.

PROPERTY 6. *Let us make a partition of the set S_n into q subsets S_{n_i}, so that the i^{th} subset, $i = 1, 2, \ldots, q$, consists of the points $u_{i1}, u_{i2}, \ldots, u_{in_i}$, with $n_1 + n_2 + \cdots + n_q = n$, and the values f_{ik_i}, $i = 1, 2, \ldots, q$; $k_i = 1, 2, \ldots, n_i$, correspond to the points u_{ik_i}. The indexing of the points in the subsets need not depend on the indexing in the set, provided the bijectivity is preserved. Then, operator (1) can be written in the form*

$$\Phi(x; f, S_n) = \sum_{i=1}^{q} \Phi(x; f, S_{n_i}) \frac{A_i}{\sum_{j=1}^{q} A_j}, \tag{7}$$

having set

$$A_i = \sum_{k_i=1}^{n_i} \frac{1}{d^{s-2}(x, u_{ik_i})}. \tag{8}$$

PROOF. Operator (1), applied to the i^{th} subset S_{n_i}, gives

$$\Phi(x; f, S_{n_i}) = \sum_{k_i=1}^{n_i} f_{ik_i} \frac{1/d^{s-2}(x, u_{ik_i})}{\sum_{h_i=1}^{n_i} (1/d^{s-2}(x, u_{ih_i}))}. \tag{9}$$

The same formula applied to the whole set S_n gives (1), but can also be written in the form

$$\Phi(x; f, S_n) = \sum_{i=1}^{q} \sum_{k_i=1}^{n_i} f_{ik_i} \frac{1/d^{s-2}(x, u_{ik_i})}{\sum_{j=1}^{q} \sum_{h_j=1}^{n_j} \left(1/d^{s-2}\left(x, u_{jh_j}\right)\right)}. \tag{10}$$

Then, formula (7) follows from (9) and (10). ∎

In particular, the given set S_n can be partitioned in two subsets only, say $S_{n_1} = \{x_1, x_2, \ldots, x_{n-1}\}$ and $S_{n_2} = \{x_n\}$; the corresponding forms of (7) and (8) are

$$\Phi(x; f, S_n) = \frac{\Phi(x; f, S_{n_1})\, A_1 + f_n\, (1/d^{s-2}(x, u_n))}{A_1 + 1/d^{s-2}(x, u_n)} \tag{11}$$

and

$$A_1 = \sum_{k=1}^{n-1} \frac{1}{d(x, u_k)}.$$

Now, formula (7) is very suitable for parallel and multistage computation, whereas (11) is shaped for recursive computation, when new data points are added one at a time (see [10,11]).

If all the nodes u_j are contained within a sphere $S \subset \mathbb{R}^s$, it is interesting to inquire into the asymptotic behavior of $\Phi(x; f, S_n)$ as x recedes indefinitely outward from S. Now we have

$$\lim_{t \to \infty} \Phi(x; f, S_n) = \frac{1}{n} \sum_{j=1}^{n} f_j, \tag{12}$$

where t is the distance from x to the sphere; namely, $\Phi(x; f, S_n)$ approaches asymptotically the average of the values f_j (see also [3]).

3. THE INTERPOLATION OPERATOR FOR CONTINUOUS DISTRIBUTIONS

Many of the considerations developed in Section 2 can be extended to the continuous case. Let us consider the s-variate integral (2) where the set $V \subset \mathbb{R}^s$ is bounded, closed, and measurable, $\mu(u)$ is stepwise continuous and bounded in V, and $d(x, u)$ is the distance of the points $u, x \in \mathbb{R}^s$. If x belongs to V, we have $d(x, u) = 0$ when u coincides with x. Hence, in this case x is a singular point of the integrand, and thus the integral is improper; precisely, it is an improper integral of second type which converges absolutely. A feature of this improper integral dependent on the parameter x is that the singular point of the integrand is equal to the parameter; one says that there is a "variable singularity". In any case, integral (2) is a continuous function of the parameter x (see, for instance, [5,8,12]).

Differentiating (2) with respect to the coordinates $(x^{(1)}, x^{(2)}, \ldots, x^{(s)})$ of $x \in \mathbb{R}^s$, we find

$$\frac{\partial U(x)}{\partial x^{(i)}} = \int_V \mu(u) \frac{u^{(i)} - x^{(i)}}{d^s(x, u)}\, du, \qquad i = 1, 2, \ldots, s. \tag{13}$$

These partial derivatives are equal to the components $F^{(i)}$ of the attraction which is exerted on a material point of mass one located at x by a mass of variable density $\mu(u)$ distributed

in the domain V. Note that for a gravitational attraction properly so-called $\mu(u)$ is essentially nonnegative; if it is a matter of electrical action, $\mu(u)$ can be positive, negative, or null. We assume $\mu(u) \geq 0$, since this is always possible, as we have seen above considering the discrete case.

We observe that the integral representation (13) of $F^{(i)}$ is included in the more general expression of integrals with a variable singularity (see, e.g., [5])

$$\int_V \frac{f(x,u)}{d^\alpha(x,u)}\,du, \qquad \alpha < s. \tag{14}$$

It is reasonable to expect that operator (2) works well also for approximating (14), but we will discuss this topic in a forthcoming paper.

PROPERTY 7. *The potential energy*

$$W(x) = q\,\frac{1}{s-2}\int_V \frac{\mu(u)}{d^{s-2}(x,u)}\,du$$

does not change if the density $\mu(u)$ is replaced by the uniform density

$$\Phi(x;\mu,V) = \frac{\int_V \left(\mu(u)/d^{s-2}(x,u)\right)\,du}{\int_V \left(1/d^{s-2}(x,u)\right)\,du}, \tag{15}$$

which depends on the parameter x, but is independent of the integration variable u.

PROOF. Reasoning as in the discrete case, the invariance property holds for a suitable constant density c, such that

$$q\,\frac{1}{s-2}\int_V \frac{\mu(u)}{d^{s-2}(x,u)}\,du = q\,\frac{1}{s-2}\int_V \frac{c}{d^{s-2}(x,u)}\,du,$$

and from this we have $c \equiv \Phi(x;\mu,V)$. In a different way, relation (15) can be obtained by writing the potential in the form

$$\begin{aligned} U(x) &= \frac{1}{s-2}\int_V \frac{\mu(u)}{d^{s-2}(x,u)}\,du\,\frac{\int_V \left(1/d^{s-2}(x,u)\right)\,du}{\int_V \left(1/d^{s-2}(x,u)\right)\,du} \\ &= \Phi(x;\mu,V)\,\frac{1}{s-2}\int_V \frac{1}{d^{s-2}(x,u)}\,du. \end{aligned} \tag{16}$$ ∎

Moreover, we observe that $\Phi(x;\mu,V)$ is uniquely determined by (15) and appears as an integral mean. It minimizes the functional

$$\int_V [\mu(u) - y]^2\,\frac{1}{d^{s-2}(x,u)}\,du, \qquad y \in \mathbb{R},$$

and can also be interpreted as a solution of an approximation problem by weighted least squares.

In order to obtain approximation (3), it is convenient to summarize the connection between discrete and continuous distributions. This is usually explained interpreting the force field of a continuous distribution as the limit case of the force field of a sequence of discrete distributions. Let us break up V into parts V_j of volume ΔV_j supporting the mass Δm_j, and choose an arbitrary point u_j in each V_j. The value of the force exerted on the unit mass at the point x by the element Δm_j, whose mass is supposed to be concentrated in u_j, is given by

$$\frac{\Delta m_j}{d^{s-1}(x,u_j)}.$$

This elementary force has the direction of the line segment joining x and u_j, and its projection on the i^{th} coordinate axis results

$$\frac{\Delta m_j}{d^{s-1}(x,u_j)} \frac{u_j^{(i)} - x^{(i)}}{d(x,u_j)}, \qquad i = 1,2,\ldots,s.$$

As a consequence, the i^{th} projection of the total force is given by the sum extended over all the elements of the partition

$$F_d^{(i)} = \sum_{j=1}^{n} \frac{u_j^{(i)} - x^{(i)}}{d^s(x,u_j)} \Delta m_j, \qquad i = 1,2,\ldots,s. \tag{17}$$

Indicating with $\mu(u)$ the mass density at the point u, we have

$$\Delta m_j \approx \mu(u_j)\, \Delta V_j$$

and from (17)

$$F_d^{(i)} \approx \sum_{j=1}^{n} \mu(u_j) \frac{u_j^{(i)} - x^{(i)}}{d^s(x,u_j)} \Delta V_j, \qquad i = 1,2,\ldots,s.$$

The limit of the sequence of these integral sums as the maximal diameter of the elements V_j, $j = 1,2,\ldots,n$, tends to zero is the integral

$$F^{(i)} = \int_V \mu(u) \frac{u^{(i)} - x^{(i)}}{d^s(x,u)}\, du, \qquad i = 1,2,\ldots,s.$$

If the point x is at a positive distance from V, the function $d(x,u)$ does not vanish and the integral is proper. On the contrary, if x belongs to V, the integrand has a singular point and the integral is improper, but it converges.

From the considerations just developed, we infer also the convergence of the sequence of the integral sums

$$\sum_{j=1}^{n} \frac{\mu(u_j)}{d^{s-2}(x,u_j)} \Delta V_j$$

to the potential function $U(x)$. Then, we can prove the following property.

PROPERTY 8. *The discrete operator $\Phi(x;\mu,S_n)$ in (1) is approximately equal to the corresponding continuous operator $\Phi(x;\mu,V)$ in (15), that is*

$$\frac{\int_V \left(\mu(u)/d^{s-2}(x,u)\right) du}{\int_V \left(1/d^{s-2}(x,u)\right) du} \approx \sum_{j=1}^{n} \mu(u_j) \frac{1/d^{s-2}(x,u_j)}{\sum_{k=1}^{n} \left(1/d^{s-2}(x,u_k)\right)}, \tag{18}$$

provided n is sufficiently large and the node distribution is not too irregular. In the limit case of a constant density $\mu(u) = \mu_0$, we have $\Phi(x;\mu_0,V) = \Phi(x;\mu_0,S_n)$.

PROOF.

(i) Let us consider a s-dimensional tesselation, whose cells have the same volume v, and the tessellated set T, which contains an entire number n of cells and is the minimal covering of V. To exemplify, we may think to the cells as s-dimensional tetrahedra. In each cell $T_h \subseteq V$ or part of cell $T_k \cap V$ a point of V is arbitrarily chosen. Then, we form the sum

$$\sum_{j=1}^{n} \mu(u_j) \frac{1}{d^{s-2}(x,u_j)} v = \sum_{j=1}^{n} \mu(u_j) \frac{1}{d^{s-2}(x,u_j)} \frac{\Delta T}{n}, \tag{19}$$

which is extended to all the cells T_j of T. The limit of the integral sum (19) as the cell diameter δ tends to zero (or the number of cells tends to infinity) is

$$\begin{aligned}\lim_{\delta\to 0}\sum_{j=1}^{n}\mu(u_j)\frac{1}{d^{s-2}(x,u_j)}v &= \lim_{n\to\infty}\sum_{j=1}^{n}\mu(u_j)\frac{1}{d^{s-2}(x,u_j)}\frac{\Delta T}{n}\\ &= \int_V \mu(u)\frac{1}{d^{s-2}(x,u)}\,du,\end{aligned} \tag{20}$$

since this integral exists (in particular, for $\mu(u)\equiv 1$). From (15) and (20), it follows for x at a positive distance from V

$$\begin{aligned}\Phi(x;\mu,V) &= \frac{\lim_{n\to\infty}\sum_{j=1}^{n}\mu(u_j)\left(1/d^{s-2}(x,u_j)\right)(\Delta T/n)}{\lim_{n\to\infty}\sum_{k=1}^{n}\left(1/d^{s-2}(x,u_k)\right)(\Delta T/n)}\\ &= \lim_{n\to\infty}\frac{\sum_{j=1}^{n}\mu(u_j)\,1/d^{s-2}(x,u_j)}{\sum_{k=1}^{n}1/d^{s-2}(x,u_k)} = \lim_{n\to\infty}\Phi(x;\mu,S_n).\end{aligned}$$

(ii) If x belongs to V (it is not an essential restriction to assume that x is an interior point of V, see [8]), the integrals in $\Phi(x;\mu,V)$ exist, but are improper, and can no more be defined as limits of integral sums. Nevertheless, by definition, the integral

$$\int_V \frac{\mu(u)}{d^{s-2}(x,u)}\,du \tag{21}$$

(in particular, for $\mu(u)\equiv 1$) exists, provided

$$\lim_{\delta\to 0}\int_{V-V_x}\frac{\mu(u)}{d^{s-2}(x,u)}\,du \tag{22}$$

exists, where V_x is any neighborhood of x subject to the sole restriction that its diameter shall not exceed δ. Hence, integral (21) can be arbitrarily approximated by the integral in (22), which is not improper.

(iii) If $x=u_j$ for any j, then the right-hand side of (18) becomes $\Phi(u_j;\mu,S_n)=\mu(u_j)$. On the other hand, the left-hand side gives

$$\inf_V \mu(u) \le \Phi(x;\mu,V) \le \sup_V \mu(u).$$

So, at worst

$$|\Phi(x;\mu,V)-\Phi(u_j;\mu,S_n)| \le \sup_V \mu(u) - \inf_V \mu(u).$$

Actually, when x coincides with a node, it is convenient to use the approximation $\Phi(x;\mu,\ S_n - \{x\})$.

(iv) For a constant density $\mu(u)=\mu_0$, we have, from Property 4(b) and the corresponding property in the continuous case, that $\Phi(x;\mu,V)=\Phi(x;\mu,S_n)$. In practice, one can say that the more slowly $\mu(u)$ varies, the more the approximation is accurate. ∎

As x recedes largely outward from V, that is, if $d(x,u)$ is very large compared with $\operatorname{diam}(V)$, we have by (3) and (12)

$$\begin{aligned}U(x) &\approx \frac{1}{n}\sum_{j=1}^{n}\mu(u_j)\,\frac{1}{s-2}\int_V\frac{1}{d^{s-2}(x,u)}\,du\\ &\approx \frac{1}{n}\sum_{j=1}^{n}\mu(u_j)\,\frac{1}{s-2}\,\frac{1}{r^{s-2}}\,\Delta V,\end{aligned}$$

where r is the distance between x and the center of mass of V. This result is close for $s = 3$ to the usual estimation

$$U(x) \approx \frac{m}{r},$$

where m is the mass of V (see, e.g., [13]).

4. REPRESENTING POTENTIAL SURFACES

Equipotential surfaces are very important to give a picture of a conservative force field. Now approximations of equipotential surfaces are given by (3) and (4), respectively, in the form

$$\Phi(x;\mu,S_n)\frac{1}{s-2}\int_V \frac{1}{d^{s-2}(x,u)}\,du \approx \text{const}, \qquad \Phi(x;U,S_n) \approx \text{const}.$$

Unfortunately, these approximations have an undesirable feature for general use in that cusps or flat spots occur at each node u_j. To avoid the drawback, it is convenient to modify (1) by using information about derivatives either given or generated from the data. This results in an approximation of the form

$$\Phi(x;L,S_n) = \sum_{j=1}^{n} L_j \frac{1/d^{s-2}(x,u_j)}{\sum_{k=1}^{n} 1/d^{s-2}(x,u_k)}, \tag{23}$$

where

$$L_j = f(u_j) + \left((\operatorname{grad} f)_{u_j}, x - u_j\right), \tag{24}$$

the last term in the right-hand side being the scalar product of the gradient of f evaluated at u_j and $x - u_j$. Relation (23), that can be rewritten as

$$\Phi(x;L,S_n) = \Phi(x;f,S_n) + \Phi\left(x;\left((\operatorname{grad} f)_{u_j}, x - u_j\right),S_n\right), \tag{25}$$

brings more information about $f(x)$ than (1). In fact, the latter considers only the value $f(u_j)$ of the interpolated function, whereas the former makes use of additional information given by the directional derivative of $f(x)$ evaluated at u_j in the direction $x - u_j$. The introduction of the linear term of the Taylor expansion in (24) is approximately equivalent to considering masses placed very near to the point u_j along certain directions. From an analytical point of view, we can consider these masses as concentrated in u_j, but we must remember their linear behavior.

If instead of (24) we consider the truncated Taylor expansion of $f(x)$ up to derivatives of order m at the point u_j

$$L_j^{(m)} = f(u_j) + df(u_j) + \cdots + d^m f(u_j), \tag{26}$$

where $d^p f(u_j)$ means the total differential of f at the point u_j and referred to the displacement $x - u_j$, we have the following extension of (24) and (25):

$$\begin{aligned}\Phi\left(x;L^{(m)},S_n\right) &= \sum_{j=1}^{n} L_j^{(m)} \frac{1/d^{s-2}(x,u_j)}{\sum_{k=1}^{n} 1/d^{s-2}(x,u_k)} \\ &= \Phi(x;f,S_n) + \Phi(x;df,S_n) + \cdots + \Phi(x;d^m f,S_n).\end{aligned}$$

Instead of the Taylor expansion (26), in particular (24), we can consider any other local approximation to the function $f(x)$ at u_j built up by using preferably functional values only; that is, the values of the function at the point u_j and at the nearest points. For this purpose, an

interpolation formula or a constrained quadratic least squares scheme can work usefully. A practical way is to get local approximants $M_j(x)$ to $f(x)$ at the points u_j, $j = 1, \ldots, n$, obtained by means of the moving weighted least-squares method using weight functions with reduced compact support. Then, the interpolating operator is expressed as a convex combination of the local approximations

$$\Phi(x; M_j, S_n) = \sum_{j=1}^{n} M_j(x) \frac{1/d^{s-2}(x, u_j)}{\sum_{k=1}^{n} 1/d^{s-2}(x, u_k)}. \tag{27}$$

It is often convenient to extend (27) in the following way:

$$\Phi(x; M_j, S_n) = \sum_{j=1}^{n} M_j(x) \frac{\tau(x, u_j)/d^{s-2}(x, u_j)}{\sum_{k=1}^{n} \tau(x, u_k)/d^{s-2}(x, u_k)}, \tag{28}$$

where $\tau(x, u)$, with $x \in \mathbb{R}^s$, $u \in D$, is a continuous positive real function. Choosing suitably $\tau(x, u)$, one can modify the weights in (28) in order either to cancel a useless characteristic or to introduce a new feature. In particular, it is possible to localize the method considering a factor $\tau(x, u)$ rapidly decreasing with distance. The formulas obtained by this way maintain, in general, the analytical and computational properties of the corresponding original ones.

A very good performance is achieved by a version of (28) which uses

$$\tau(x, u_j) = \left(1 - \frac{d(x, u_j)}{\rho_j}\right)_+^2,$$

where ρ_j is the radius of the circle of support at the point x_j, and $(u)_+ > 0$ if $u > 0$, $(u)_+ = 0$ if $u \leq 0$. This method has been developed by Franke and Nielson [14], and Renka [15] for Shepard's operator.

5. OUTLINE OF COMPUTATIONAL PERFORMANCES

Computational performance of $\Phi(x; f, S_n)$ in (1) and its modified forms in Section 4 is sufficiently known, because either these methods have been tested directly or their behavior can be inferred from analogy. To be more precise, Franke [16] tested many programs for scattered data interpolation in $\mathbb{R}^2$ and, in particular, a modification of $\Phi(x; f, S_n)$, named *modified quadratic Shepard method* (see [17]), where $S_n \subset \mathbb{R}^2$ and the power of distance is $d^2(x, u_j)$. To enable testing different methods in a consistent manner, Franke developed a set of standard subprograms which generate the test cases, compute deviation statistics, and generate and label perspective plots of the surfaces. Typically subprograms generate the values of the interpolant at the vertices of a rectangular mesh. Then other authors (see, e.g., [18,19]) surveyed the interpolation to trivariate functions and the construction of surfaces in four-dimensional Euclidean space. Among other methods $\Phi(x; f, S_n)$, for $S_n \subset \mathbb{R}^3$ and $d^2(x, u_j)$, or modified forms were considered again. Summing up, the computational performance of $\Phi(x; U, S_n)$ and $\Phi(x; \mu, S_n)$, when they are used for interpolating potential and for remapping density or potential, is the same performance of $\Phi(x; f, S_n)$ in general.

Methods that depend on metrics, as (1) and Hardy's multiquadrics, are easily generalized to higher-dimensional spaces and their performance comes up to expectation. Nevertheless, these methods are global and give more acceptable results for small data sets, but for larger sets (from some hundred points up to some millions), likely to be met in practice, they tend to become computationally too expensive. If very large sets of data were to be considered, it is clear that a different implementation approach might be necessary. An efficient solution to this problem is given by the form (7) of $\Phi(x; f, S_n)$. It is particularly suitable for parallel computation and, under the condition of a well-balanced workload, the speed up factor is approximately equal to

the number of processors involved. Moreover, equation (7) can be used to evaluate $\Phi(x; f, S_n)$ in subsequent stages, enlarging as many times as necessary the data set, without repeating each time the whole computation. In particular, a recursive computation of $\Phi(x; f, S_n)$ can be done by using (11).

To test exhaustively the numerical performance of (3) is a hard task. In fact, on one hand, we need to consider those particular cases of integral (2) whose primitives are known, but these are very few; on the other hand, it is significant to compare the performance of (3) with that of other methods used in practice, but this goes beyond the limits of this paper. As an example, for the evaluation of the gravitational potential ($s = 3$), it is a standard procedure to divide the integration set V into a collection of rectangular prisms or a stack of thin laminas of constant density (see [20] and the references cited therein). Hence, we limit ourselves to consider the typical case of the volume potential of a mass distributed inside a sphere $V \subset \mathbb{R}^3$ of radius r with variable density $\mu(u)$, which depends only on the distance of u from the centre of the sphere. This example is interesting in itself because it deals with a spherical approximation of the Earth that is adequate for solving many problems (as for example, volume fitting of scattered data inside the crust of the Earth (see, e.g., [6])).

It is convenient to use spherical coordinates (ρ, θ, ϕ) and to identify the polar axis with the straight line from the center of the sphere to the point x (see, for instance, [21]). With these notations, the density can be written $\mu(u) = \nu(\rho)$ and the potential

$$U(x) = \begin{cases} \dfrac{4\pi}{|x|} \int_0^r \nu(\rho)\rho^2 \, d\rho = \dfrac{M(r)}{|x|}, & \text{if } |x| \geq r, \\ \dfrac{4\pi}{|x|} \int_0^{|x|} \nu(\rho)\rho^2 \, d\rho + 4\pi \int_{|x|}^r \nu(\rho)\rho \, d\rho, & \text{if } |x| \leq r, \end{cases}$$

where $|x|$ is the length of the vector x and $M(r)$ the total mass. If, in particular, $\nu(\rho) = \nu_0$ inside the sphere of radius r, from the general formula we obtain

$$U(x) = \begin{cases} \dfrac{4\pi}{3|x|} \nu_0 r^3, & \text{if } |x| \geq r, \\ 2\pi\nu_0 \left(r^2 - \dfrac{|x|^2}{3} \right), & \text{if } |x| \leq r. \end{cases}$$

If the mass density $\nu(\rho)$ is decreasing with ρ, it is always possible to get $\nu(r) = 0$; that is, the mass density reduces to zero on the boundary of the sphere. In fact, if $\nu(r) > 0$, we can consider the potential function

$$U(x; \nu(\rho) - \nu(r)) = U(x; \nu(\rho)) - U(x; \nu(r)),$$

where $U(x; \nu(\rho)) \equiv U(x)$ is given by (2) and the others have a similar meaning. As a consequence, the integration domain V for $U(x)$ can be extended to a cube $\tilde{V}$, circumscribed to the considered sphere, where the mass density is continuous in the cube but equals zero outside the sphere. Moreover, a translation and a scale change maps the cube $\tilde{V}$ in the unit cube V^* in $\mathbb{R}^3$. It permits us to use a standard algorithm proposed by Renka [15]. We considered 1280 nodes, generated by a random uniform distribution, and a linear expression of $\nu(\rho)$.

A lot of calculations showed that the relative error of approximation (3), which depends on the parameter x, on the smoothness of the density $\nu(\rho)$, and on the distribution of the nodes, may vary from zero to rather large values, comparable to errors of the basic Monte Carlo method. In conclusion, we can affirm for now that approximation (3) is interesting mainly for working on scattered data in higher dimensions, as well as for possibility of parallel, multistage, and iterative computation. In a forthcoming paper, the proposed method will be applied to some interesting models and extensively tested from a numerical viewpoint.

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